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PROPERLY IMMERSED MINIMAL SURFACES IN A SLAB OF $\mathbb{H} \times \mathbb{R}$, $\mathbb{H}$ THE HYPERBOLIC PLANE

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Abstract. We prove that the ends of a properly immersed simply or one connected minimal surface in $\mathbb{H} \times \mathbb{R}$ contained in a slab of height less than π of $\mathbb{H} \times \mathbb{R}$, are multi-graphs. When such a surface is embedded then the ends are graphs. When embedded and simply connected, it is an entire graph.

1. Introduction

A fundamental problem in surface theory is to understand surfaces of prescribed curvature in homogenous 3-manifolds. Simply connected properly embedded surfaces are the simplest to consider and after the compact sphere, the plane is next. There are some unicity results. A proper minimal embedding of the plane in $\mathbb{R}^3$ is a flat plane or a helicoid [11]. A proper embedding of the plane in hyperbolic 3-space as a constant mean curvature one surface (a Bryant surface) is a horosphere [4]. Also, a proper minimal embedding of an annulus $\mathbb{S}^1 \times \mathbb{R}$ in $\mathbb{R}^3$ is a catenoid [3] and a proper minimal embedding of an annulus with boundary,-$\mathbb{S}^1 \times \mathbb{R}^+$-, is asymptotic to an end of a catenoid, plane or helicoid [11, 1]. This is true for Bryant annular ends in $\mathbb{H}^3$. A proper constant mean curvature one embedding of an annulus $\mathbb{S}^1 \times \mathbb{R}$ in $\mathbb{H}^3$ is a catenoid cousin. Such an annulus with compact boundary in $\mathbb{H}^3$ is asymptotic to an end of a catenoid cousin or a horosphere.

In this paper we consider proper minimal embeddings (and immersions) of the plane and the annulus in $\mathbb{H} \times \mathbb{R}$.

Contrary to $\mathbb{R}^3$, there are many such surfaces in $\mathbb{H} \times \mathbb{R}$. Given any continuous rectifiable curve $\Gamma \subset \partial_\infty(\mathbb{H} \times \mathbb{R})$, Γ a graph over $\partial_\infty(\mathbb{H})$, there is an entire minimal graph asymptotic to Γ at infinity [12, 13]. Also if Γ is an ideal polygon of $\mathbb{H}$, there are necessary and sufficient conditions on Γ which ensure the existence of a minimal graph over the interior of Γ, taking values plus and minus infinity on alternate sides of Γ [5]. This graph is then a simply connected minimal surface in $\mathbb{H} \times \mathbb{R}$. We will see there are many minimal embeddings of the plane that are not graphs (aside from the trivial example of a (geodesic of $\mathbb{H}$) $\times \mathbb{R}$; a vertical plane.

In this paper we will give a condition which obliges a properly embedded minimal plane to be an entire graph. More generally, we will give a condition which obliges properly immersed minimal surfaces of finite topology to have multi-graph ends.

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We will see that slabs S of height less than π play an important role. There are two reasons for this. First, complete vertical rotational catenoids exist precisely when their height is less than π. Secondly, for $h > \pi$, there are vertical rectangles of height h at infinity; i.e., in $\partial_\infty(\mathbb{H}\times\mathbb{R})$, that bound simply connected minimal surfaces $M(h)$ (the rectangle is the asymptotic boundary of $M(h)$). These surfaces are invariant by translation along a horizontal geodesic and we discuss them in detail later; [9, 14, 15].

Let $\epsilon > 0$ and $S = \{(p,t) \in \mathbb{H}\times\mathbb{R}; |t| \leq (\pi-\epsilon)/2\}$. We will prove that simply or one-connected properly immersed minimal surfaces in $\mathbb{H}\times\mathbb{R}$, $\Sigma \subset S$ have multi-graph ends. In fact, an annular properly immersed minimal surface in S, with compact boundary, has a multi-graph subend. More precisely we prove:

Slab Theorem 1. *Let $S \subset \mathbb{H}\times\mathbb{R}$ be a slab of height $\pi-\epsilon$ for some $\epsilon > 0$. Assume Σ is a properly immersed minimal surface in $\mathbb{H}\times\mathbb{R}$, $\Sigma \subset S$. If Σ is simply connected and embedded then Σ is an entire graph.*

More generally;

(1) *If Σ is of finite topology and embedded with one end then Σ is simply connected and an entire graph.*
(2) *If Σ is homeomorphic to $\mathbb{S}^1\times\mathbb{R}^+$, then an end of Σ is a multi-graph. If this annular surface is embedded then an end is a graph.*

In particular, by (2), if Σ is of finite topology then each end of Σ is a multi-graph.

Results of this nature have been obtained by Colding and Minicozzi in their study of embedded minimal disks in balls of $\mathbb{R}^3$, whose boundary is on the boundary of the ball; see proposition III. 1.1 of [2]. We have been inspired here by their ideas; in particular using foliations by catenoids to control minimal surfaces.

Remark 1.1. *We will give an example of an Enneper-type minimal surface in S. This is a properly immersed minimal surface in $\mathbb{H}\times\mathbb{R}$, $\Sigma \subset S$ that is simply connected and whose end is a 3-fold covering graph over the complement of a compact disc in $\mathbb{H}$.*

Remark 1.2. *Also there is an example of a properly embedded simply connected minimal surface in a slab of height π that is not a graph. We will describe this surface after the proof of the Slab Theorem. Thus π is optimal for the slab theorem.*

2. The Dragging Lemma

Dragging Lemma 1. *Let $g : \Sigma \to N$ be a properly immersed minimal surface in a complete 3-manifold N. Let A be a compact surface (perhaps with boundary) and $f : A\times[0,1] \to N$ a $\mathcal{C}^1$-map such that $f(A\times\{t\}) = A(t)$ is a minimal immersion for $0 \leq t \leq 1$. If $\partial(A(t)) \cap g(\Sigma) = \emptyset$ for $0 \leq t \leq 1$ and $A(0)\cap g(\Sigma) \neq \emptyset$, then there is a $\mathcal{C}^1$ path $\gamma(t)$ in Σ, such that $g\circ\gamma(t) \in A(t)\cap g(\Sigma)$ for $0 \leq t \leq 1$. Moreover we can prescribe any initial value $g\circ\gamma(0) \in A(0)\cap g(\Sigma)$.*

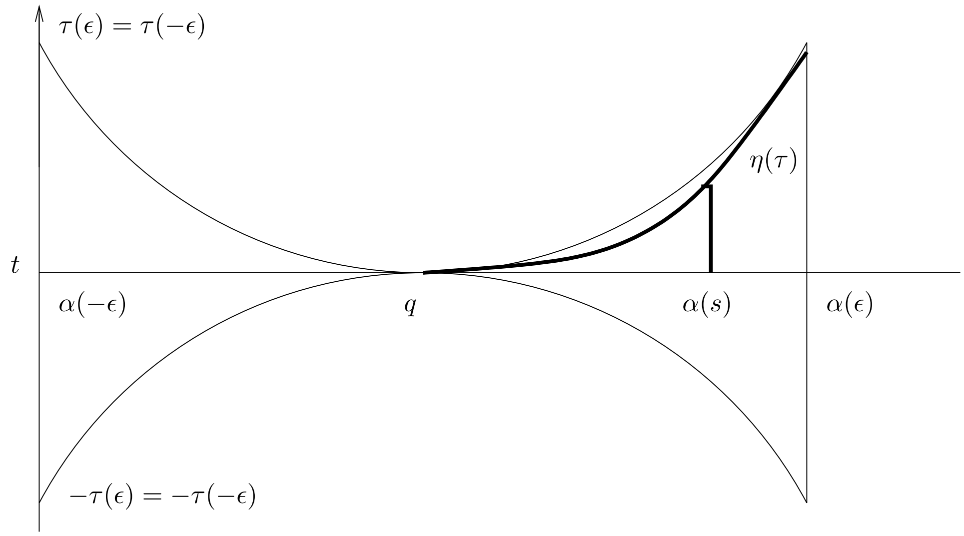

FIGURE 1. Neighborhood of a singular point

Remark 2.1. *To obtain a $\gamma(t)$ satisfying the Dragging lemma that is continuous (not necessarily $\mathcal{C}^1$) it suffices to read the following proof up to (and including) Claim 1.*

Proof. When there is no chance of confusion we will identify in the following Σ and its image $g(\Sigma)$, $\gamma\subset\Sigma$ and $g\circ\gamma$ in $g(\Sigma)\subset N$. In particular when we consider embeddings of Σ there is no confusion.

Let $\Sigma(t)=g(\Sigma)\cap A(t)$ and $\Gamma(t)=f^{-1}(\Sigma(t))$, $0\le t\le 1$ the pre-image in $A\times[0,1]$.

When $g:\Sigma\to N$ is an immersion, we consider $p_0\in g(\Sigma)\cap A(0)$, and pre-images $z_0\in g^{-1}(p_0)$ and $(q_0,0)\in f^{-1}(p_0)$. We will obtain the arc $\gamma(t)\in\Sigma$ in a neighborhood of z_0 by a lift of an arc $\eta(t)$ in a neighborhood of $(q_0,0)$ in $\Gamma([0,1])$ i.e. $g\circ\gamma(t)=f\circ\eta(t)$. We will extend the arc continuously by iterating the construction.

Since $\Gamma(t)$ represents the intersection of two compact minimal surfaces, we know $\Gamma(t)$ is a set of a finite number of compact analytic curves $\Gamma_1(t),...,\Gamma_k(t)$. These curves $\Gamma_i(t)$ are analytic immersions of topological circles. By hypothesis, $\Gamma(t)\cap(\partial A\times[0,1])=\emptyset$ for all t. The maximum principle assures that the immersed curves can not contain a small loop, nor an isolated point. Since $A(t)$ is compact and has bounded curvature, a small loop in $\Gamma(t)$ would bound a small disc D in Σ with boundary in A. Since A is locally a stable surface, we can consider a local foliation around the disc and find a contradicttion with the maximum principle. We say in the following that $\Gamma(t)$ does not contain small loops.

Claim 1: We will see that for each t with $\Gamma(t)\neq\emptyset$, $t<1$ there is a $\delta(t)>0$ such that if $(q,t)\in\Gamma(t)$, then there is a $\mathcal{C}^1$ arc $\eta(\tau)$ defined for $t\le\tau\le t+\delta(t)$ such that $\eta(t)=(q,t)$ and $\eta(\tau)\in\Gamma(\tau)$ for all τ (there may be values of t where $\gamma'(t)=0$).

Since $\Gamma(0) \neq \emptyset$, this will show that the set of t for which $\eta(t)$ is defined is a non empty open set. This defines an arc $\gamma(\tau)$ as a lift of $f \circ \eta(\tau) \subset A(\tau)$ in a neighborhood of $\gamma(t) \in \Sigma$.

First suppose $(q,t) \in \Gamma(t)$ is a point where $A(t) = f(A \times \{t\})$ and $g(\Sigma)$ are transverse at $f(q,t)$. Let us consider the $\mathcal{C}^1$ immersions

$$F : A \times [0,1] \to N \times [0,1] \text{ with } F(q,t) = (f(q,t),t)$$

$$G : \Sigma \times [0,1] \to N \times [0,1] \text{ with } G(z,t) = (g(z),t).$$

Let $\hat{M} = F(A \times [0,1]) \cap G(\Sigma \times [0,1])$ and $M = F^{-1}(\hat{M})$. $F(A \times [0,1])$ and $G(\Sigma \times [0,1])$ are transverse at $p = F(q,t)$. Thus $\hat{M}$ is a 2-dimensional surface of $N \times [0,1]$ near p . We consider $X(t)$ a tangent vector field along $\Gamma(t)$ and $JX(t)$ an orthogonal vector field to $X(t)$ in $T_{(q,t)}M$. If $\partial/\partial t \perp T_p\hat{M}$, then $T_p\hat{M} = T_{f(q,t)}A(t) = T_{f(q,t)}g(\Sigma)$ and (q,t) would be a non transverse point of intersection of $A(t)$ and $g(\Sigma)$. Thus $< JX(t), \partial/\partial t > \neq 0$ and we can find $\eta(\tau)$ a smooth path, defined for $\tau \in [t-\delta(q), t+\delta(q)]$ such that $\eta(t) = (q,t)$ and $\eta'(t) = JX(t)$ is transverse to $\Gamma(t)$ at (q,t).

By transversality and f being $\mathcal{C}^1$ in the variable t, we have a $\delta(q) > 0$ such that for $t - \delta(q) \leq \tau \leq t + \delta(q)$, $A(\tau)$ intersects $f \circ \eta(\tau)$ in a unique point and this point varies continuously with $t - \delta(q) \leq \tau \leq t + \delta(q)$. With a fixed initial point in Σ, a lift of $f \circ \eta(\tau)$, defines $\gamma(\tau) \in \Sigma$.

Again by transversality, we can find a neighborhood of (q,t) in $\Gamma(t)$ and a $\delta > 0$ so that the above path $\gamma(\tau)$ exists for $t - \delta \leq \tau \leq t + \delta$, through each point in the neighborhood of q. It suffices, to look for a local immersion of a neighborhood of 0 in T_pM into M, to obtain a $\mathcal{C}^1$ diffeomorphism $\psi : B(0) \subset T_pM \to M$. M has the structure of a $\mathcal{C}^1$ manifold in a neighborhood of points of transversality and this structure extends to $F^{-1}(M) \subset A \times [0,1]$.

We will find a $\delta > 0$ that works in a neighborhood of a singular point $(q,t) \in \Gamma(t)$, where there is a $z \in \Sigma$ such that $f(q,t) = g(z)$ and $T_{f(q,t)}A(t) = T_{g(z)}g(\Sigma)$. We consider singularities of $\Gamma(t)$ where $A(t)$ and $g(\Sigma)$ are tangent. Near a singularity $(q,t) \in \Gamma(t)$, $\Gamma(t)$ contains $2k$ analytic curves intersecting at q at equal angles, $k \geq 1$.

Let V be a neighborhood of q in A. The set $\Gamma(t) \cap V$ is $2k$ analytic curves. Let $\alpha :]-\epsilon, \epsilon[\to V \cap \Gamma(t)$ be a regular parametrization of one curve with $\alpha(0) = q$ and $\alpha(\pm\epsilon) \in \partial V$. By transversality as discussed in the previous paragraph $< JX(t), \partial/\partial t > \neq 0$ at $\alpha(s)$ for $s \neq 0$ and $JX(t)$ can be integrated as a curve on M for $t - \delta(s) \leq \tau \leq t + \delta(s)$. Here $\delta(s)$ is a $\mathcal{C}^1$ function which can be chosen increasing with $\delta(0) = \delta'(0) = 0$.

There exists a $\mathcal{C}^1$ diffeomorphism $\phi : \Omega = \{(s,\tau) \in \mathbb{R}^2; -\epsilon \leq s \leq \epsilon, t - \delta(s) \leq \tau \leq t + \delta(s)\} \to M$ such that $\phi(s,t) = \alpha(s)$ for $s \in]-\epsilon, \epsilon[$ and $\phi(s,\tau) \in \Gamma(\tau)$ for $t - \delta(s) \leq \tau \leq t + \delta(s)$. We consider a function $\tau :]-\epsilon, \epsilon[\to \mathbb{R}$, such that $(s, \pm\tau(s)) \in \Omega$ and τ is increasing, $\tau(0) = \tau'(0) = 0$ and $\tau(\epsilon) = t + \delta(\epsilon)$, $\tau(-\epsilon) = t + \delta(-\epsilon)$.

Now we can construct a path $\eta(\tau) \in \Gamma(\tau)$ which joins (q,t) to a point in $\Gamma(t + \delta(\epsilon))$. The $\mathcal{C}^1$ arc $f \circ \eta(\tau), t \leq \tau \leq t + \delta(\epsilon)$ is locally parametrized by $\phi(s, \tau(s)), s \in]0, \epsilon[$ and continuously extends to $f(q,t)$ when $\tau \to t$. Each point $\alpha(s)$, can be connected $\mathcal{C}^1$,

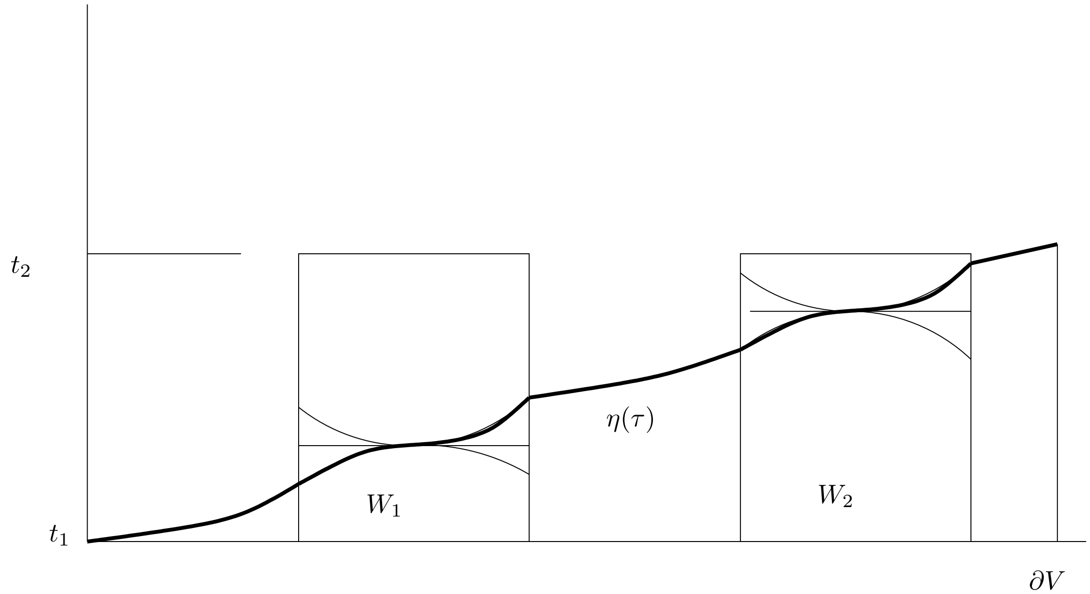

FIGURE 2. The curve $\eta(\tau)$ passing through several singularities.

by the arc $\phi(s,\tau), t \leq \tau \leq \tau(s)$ from $\alpha(s)$ to $\phi(s,\tau(s))$, and next a subarc of $\eta(\tau)$ for $\tau(s) \leq \tau \leq t + \delta(\epsilon)$ (see figure 1). The constant $\delta(\epsilon)$ depends only on $\alpha(\epsilon) = q_1$, and we note $\delta(q_1) := \delta(\epsilon)$.

Now there are a finite number of arcs α in $V - (q)$, with end points q and a collection of $q_1, q_2, ...q_{2k}$. So one has a $0 < \delta$ with $\delta < \delta(q_i)$ that works in a neighborhood of q. The claim is proved.

To complete the proof of the Dragging Lemma, it suffices to prove that $\gamma(t)$ extends $\mathcal{C}^1$ for any value of $t \in [0,1]$. Assume that there is a point t_0 such that the arc $\gamma(t)$ is defined in a $\mathcal{C}^1$ manner for $t < t_0$. By compactness of A, the arc accumulates at a point $(q, t_0) \in \Gamma(t_0)$. Remark that the structure of M along $\Gamma(t_0)$ gives easily the existence of a continuous extension to t_0. To ensure a $\mathcal{C}^1$ path through t_0, we need a more careful analysis at (q, t_0).

Claim 2: Suppose the path $\gamma(t)$ satisfies the conditions of the Dragging lemma for $0 \leq t \leq t_0 < 1$. Then $\gamma(t)$ can be extended to $0 < t < t_0 + \delta$, to be $\mathcal{C}^1$ and satisfy the conditions of the Dragging lemma, for some $\delta > 0$.

If (q, t_0) is a transversal point, M has a structure of a manifold and if $t_0 - \delta(t_0) < t_1 < t_0$ and $\eta(t_1) = (q_1, t_1)$ is in a neighborhood of (q, t_0), we can find a $\mathcal{C}^1$ arc that joins $\eta(t_1)$ to $(q, t_0) \in \Gamma(t_0)$. Next we extend the arc for $t_0 \leq t \leq t_0 + \delta(t_0)$.

If (q, t_0) is a singular point, we consider a neighborhood $V \subset A$ of q and $\Gamma(t_0)$ intersects ∂V in $2k$ transversal points $q_1, ..., q_{2k}$. We consider $V \times [t_1, t_0]$ with $t_0 - \delta(t_0) < t_1 < t_0$. By transversality at $(q_1, t_0), ..., (q_{2k}, t_0)$, the analytic set $\Gamma(t_1)$ intersects ∂V in $2k$ points and V in k analytical arcs $\alpha_1, ..., \alpha_k$. We suppose that $\eta(t_1) \in \alpha_1 \subset V \times \{t_1\}$. We construct below a monotonous $\mathcal{C}^1$ arc from $\eta(t_1)$ to a point $(\hat{q}, t_2)$ on $\partial V \times \{t_2\}$

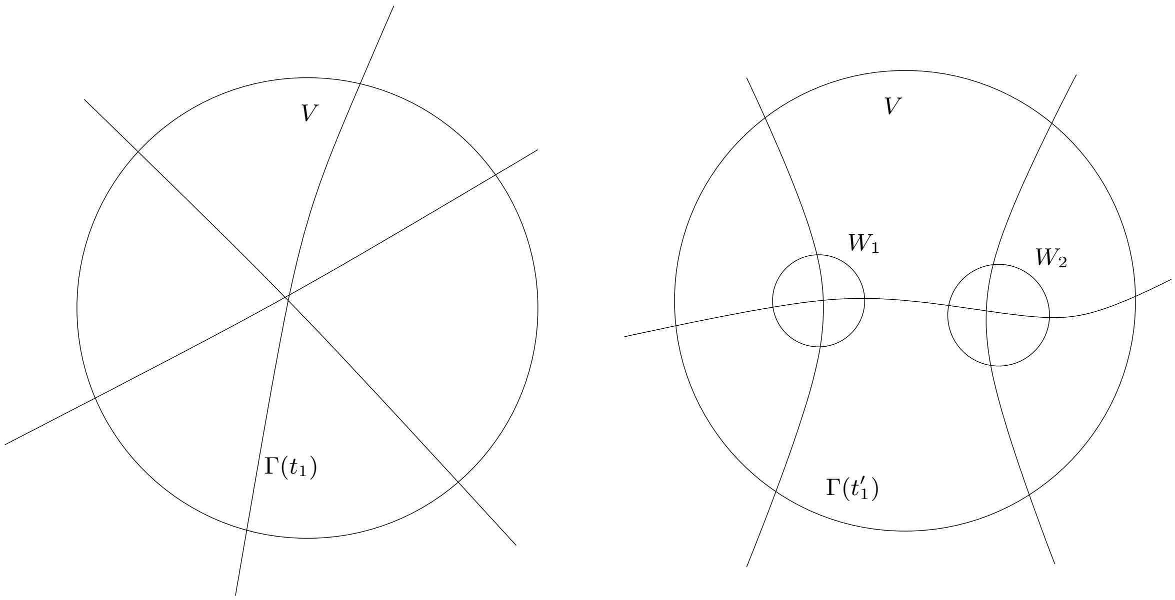

FIGURE 3. Left: The curve $\Gamma(t_1)$-Right: The curve $\Gamma(t_1')$.

for some $t_1 < t_2 < t_0$ and by transversality an arc from $(\hat{q}, t_2)$ to a point $(q', t_0) \in \partial V \times \{t_0\}$, using the fact that $t_0 - \delta(t_0) < t_2$. Next we can extend the arc in a $\mathcal{C}^1$ manner from (q', t_0) to some point in $\Gamma(t_0 + \delta(t_0))$.

We consider $(\tilde{q}_1, t_1), ..., (\tilde{q}_\ell, t_1)$ singular points of $\Gamma(t_1) \cap V \times \{t_1\}$ and we denote by $W_1, ..., W_\ell$ neighborhoods of $\tilde{q}_1, ..., \tilde{q}_\ell$ in $A \cap V$. The arc α_1 cannot have double points in V without creating small loops. Hence α_1 passes through each $W_1, ..., W_\ell$ at most one time, before joining a point of ∂V (We can restrict V in such a way that there are no small loops in V).

First we assume that there is t_2 such that for any $t \in [t_1, t_2]$, the curve $\Gamma(t)$ has exactly one isolated singularity in each neighborhood $W_i \times \{t\}$ with the same type as $\tilde{q}_i \in \Gamma(t_1)$ $(i = 1, ..., \ell)$ and $t_2 < t_1 + \delta(t_1)$. If we parametrize $\alpha_1 : [s_0, s_{2\ell+1}] \to \Gamma(t_1)$, we can find $s_1, ..., s_{2\ell}$ such that $\alpha_1(s_{2k-1}), \alpha_1(s_{2k}) \in \partial W_k$ and $I_k = [s_{2k-2}, s_{2k-1}]$ are intervals parametrizing transversal points in $\Gamma(t_1)$.

The manifold structure of M gives an immersion $\psi_j : I_j \times [t_1, t_1 + \delta] \to M$, $t_1 + \delta < t_2$ and $j = 1, ..., \ell + 1$. In the construction of η up to t_1, the singular points are isolated; then we can assume $\eta(t_1)$ is a regular point of $\Gamma(t_1)$, hence is contained in an $\alpha_1(I_j)$. We construct the beginning of the arc $\eta(\tau)$ as the graph parametrized by $\phi_j(s, \tau(s))$ with τ an increasing function from t_1 to $t_1 + \delta/n$ as s varies from $\hat{s} \in I_j$, corresponding to the initial point $\eta(t_1) = \alpha_1(\hat{s})$, to s_{2j-1}. Next we pass through the singularity $(\tilde{q}_j, t_1 + 2\delta/n)$ by constructing an arc wich joins the point $\phi_j(s_{2j-1}, t_1 + \delta/n) \in \Gamma(t_1 + \delta/n) \cap \partial W_j$ to the point $\phi_{j+1}(s_{2j}, t_1 + 3\delta/n) \in \Gamma(t_1 + 3\delta/n) \cap \partial W_j$ (see figure 2). For a suitable value of n we can iterate this construction, passing through the singularities

$\tilde{q}_j, \tilde{q}_{j+1}...$, until we join a point $(\hat{q}, t_2)$ of $\partial V \times \{t_2\}$ and then we extend the arc up to t_0 by transversality outside V.

Now we look for this interval $[t_1, t_2]$. Let $t_1 < t_1' < t_0$ and $\Gamma(t_1')$ have several singularities in some neighborhood W_k, or a unique singularity of index less the one of the $\tilde{q}_k$. We consider in this W_k a finite collection of neighborhoods of isolated singularities $W'_{k,1}, ...W'_{k,\ell'}$. We observe, by transversality that there are the same number of components of $\Gamma(t_1)$ and $\Gamma(t_1')$ in W_k (see figure 3). Hence each $W'_{k,j}$ contains a number of components of $\Gamma(t_1')$ strictly less than the number of components of $\Gamma(t_1)$ in W_k. The index of the singularity is strictly decreasing along this procedure. We can iterate this analysis up to a point where each singularity can not be reduced to a simple one. This gives the interval $[t_1, t_2]$. $\square$

3. Proof of The Slab theorem

Assume Σ is either simply connected ($\partial\Sigma = \emptyset$) or Σ is an annulus with compact boundary: Σ homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^+$. Also assume Σ is properly minimally immersed in $\mathbb{H}\times\mathbb{R}$ and $\Sigma \subset S = \{(p,t) \in \mathbb{H}\times\mathbb{R}; |t| < (\pi-\epsilon)/2\ \}$.

We fix $h \geq 2\pi$ sufficiently large so that there are points of Σ in the geodesic ball B of radius h of $\mathbb{H}\times\mathbb{R}$ with center at a point p_0 in $\mathbb{H}\times\{0\}$, and $\partial\Sigma \subset B$ (if $\partial\Sigma \neq \emptyset$).

Let $\mathrm{Cat}(p_0)$ denote a compact part of a rotational catenoid, a bi-graph over $t = 0$, bounded by two circles outside of the slab S, $p_0 \in \mathrm{Cat}(p_0)$. We assume h is suficientlly large so that B also contains $\mathrm{Cat}(p_0)$; see figure 4.

Since Σ is properly immersed, the set $B\cap\Sigma$ has a finite number of compact connected components and there is a compact K in $\mathbb{H}\times\mathbb{R}$, such that any two points of Σ in B can be joined by a path of Σ in K.

Now suppose $p \in \Sigma - K$ has a vertical tangent plane and let $M = \alpha\times\mathbb{R}$ be tangent to Σ at p, α a geodesic of $\mathbb{H}$. We will prove M must intersect K. Suppose this were not the case. M separates $\mathbb{H}\times\mathbb{R}$ in two components $M(+)$ and $M(-)$; assume $K \subset M(+)$. The local intersection of M and Σ near p consists of $2k$ curves through p, $k \geq 1$, meeting at equal angles π/k at p.

Let $\Sigma_1(+)$ and $\Sigma_2(+)$ be distinct local components at p of $\Sigma - M$, that are contained in $M(+)$. Observe that $\Sigma_1(+)$ and $\Sigma_2(+)$ are contained in distinct components of $\Sigma\cap M(+)$. Otherwise we could find a path α_0 in $\Sigma\cap M(+)$, joining a point $x \in \Sigma_1(+)$ to a point $y \in \Sigma_2(+)$. Then join x to y by a local path β_0 in Σ going through p, but $\beta_0 \subset M(+)$ except at p; see figure 5.

Let $\Gamma = \alpha_0 \cup \beta_0 \subset \overline{M(+)}$. If Σ is simply connected Γ bounds a compact disk D in Σ. If Σ is an annulus there are two cases. $\Gamma\cup\partial\Sigma$ bounds an immersed compact annulus (we also call D) or Γ bounds a compact disk in the annulus . By construction of Γ, D contains points in $M(-)$. But D is compact and minimal, $\partial D \subset \overline{M(+)}$, so there would be an interior point of D that is furthest away from M in $M(-)$. This contradicts the maximum principle.

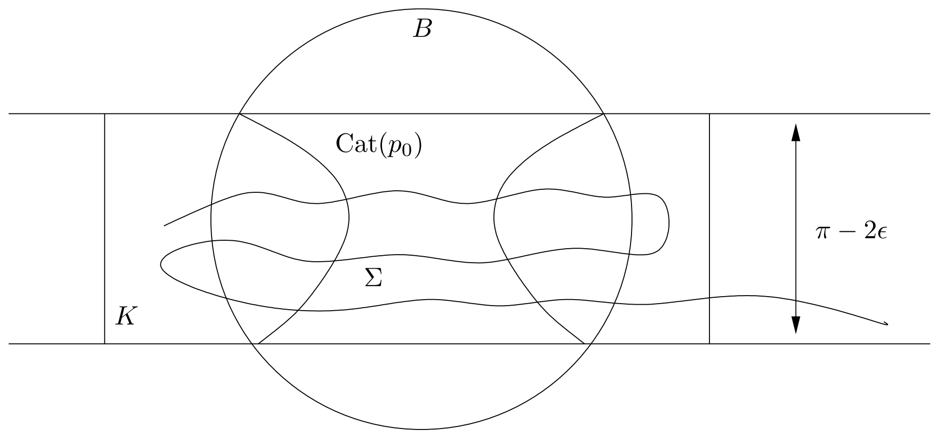

FIGURE 4. The compact K and the catenoid Cat(p_0)

Thus $\Sigma_1(+)$ and $\Sigma_2(+)$ are in distinct components Σ_1 and Σ_2 of $\Sigma \cap M(+)$. Now let $\mu(\epsilon)$ be the geodesic of length ϵ starting at p, normal to M at p, and contained in $M(+)$. We will now also denote by α, the geodesic α translated vertically to pass through p. Let $\alpha(\epsilon)$ be the complete geodesic of $\mathbb{H}$ obtained from α by translating α along $\mu(\epsilon)$ to the endpoint of $\mu(\epsilon)$ distinct from p. The distance between α and $\alpha(\epsilon)$ diverges at infinity; the two geodesics have distinct end points at infinity.

Choose ϵ small so that $M(\epsilon) = \alpha(\epsilon) \times \mathbb{R}$ meets both $\Sigma_1(+)$ and $\Sigma_2(+)$, at points $x \in \Sigma_1, y \in \Sigma_2$. If Σ is simply connected then no connected component of $M(\epsilon) \cap \Sigma$ can be compact. Hence x and y are in non-compact components $C(x) \subset \Sigma_1 \cap M(\epsilon)$ and $C(y) \subset \Sigma_2 \cap M(\epsilon)$.

We claim this also holds when Σ is an annulus; more precisely:

Claim: If Σ is an annulus then the components $C(x)$ of x in $\Sigma_1 \cap M(\epsilon)$ and $C(y)$ of y in $\Sigma_2 \cap M(\epsilon)$ are both non compact.

Proof of the Claim. First suppose $C(x)$ and $C(y)$ are compact. Neither can be null homotopic in Σ by the maximum principle, hence $C(x) \cup C(y)$ bound an immersed compact annulus D in Σ. But $\partial D \subset M(\epsilon)$ which is impossible.

It remains to show one of the $C(x), C(y)$ can not be non compact. Suppose $C(x)$ is not compact and $C(y)$ is compact. So $C(y) \cup \partial\Sigma$ bound an immersed annulus D in $M(\epsilon)(+)$. The distance between M and $M(\epsilon)$ diverges and Σ is proper so there are points z of $C(x)$ arbitrarily far from M.

Choose such a z so that one can place a vertical catenoid Cat(z) in $M(+)$ which is a horizontal translation of Cat(p_0) and contains z. The boundary $\partial(\text{Cat}(z)) \cap S = \emptyset$. Let η be a geodesic joining z to the point $p_0 \in B$. Apply the Dragging Lemma to the translation of Cat(z) along η from z to p_0, so that the translation of Cat(z) is contained in B, at the end of the movement.

We obtain a path in $\Sigma \cap M(+)$ joining z to a point $\emptyset$ of Σ in B. Then join $\emptyset$ to a point of $\partial\Sigma$ in $\Sigma \cap K$. This path contradicts that Σ_1 and Σ_2 are in distinct components of

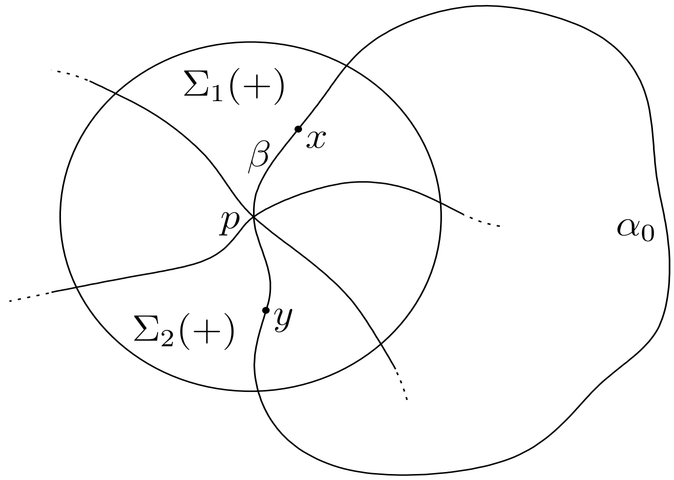

FIGURE 5. A loop Γ

$\Sigma - M$. Hence the claim is proved and both $C(x) \subset \Sigma_1 \cap M(\epsilon)$ and $C(y) \subset \Sigma_2 \cap M(\epsilon)$ are non compact.

We now continue the proof that M must intersect K. Σ is either simply connected or an annulus and we have two non compact components $C(x) \subset \Sigma_1 \cap M(\epsilon), C(y) \subset \Sigma_2 \cap M(\epsilon)$ and Σ_1, Σ_2 are distinct components of $\Sigma - M$.

Now, as we argued in the proof of the claim above , we can take points $z_1 \in C(x), z_2 \in C(y)$, far enough away from M so that one can place compact vertical catenoids $\mathrm{Cat}(z_1)$ and $\mathrm{Cat}(z_2)$, so $z_1 \in \mathrm{Cat}(z_1)$, $z_2 \in \mathrm{Cat}(z_2)$, $\partial\mathrm{Cat}(z_i) \cap S = \emptyset, i = 1, 2$ and $\mathrm{Cat}(z_i)$ are symmetric with respect to $t = 0$. Also $\mathrm{Cat}(z_i) \subset M(+), i = 1, 2$.

Take horizontal geodesics $\eta_1, \eta_2 \subset M(\epsilon)(+)$ from $\mathrm{Cat}(z_1)$ to p_0 and from $\mathrm{Cat}(z_2)$ to p_0. Apply the Dragging Lemma along η_1, η_2, to find a path in Σ from z_1 to a point $\omega_1 \in \Sigma \cap B$, and another path from z_2 to $\o_2 \in \Sigma \cap B$. Join $\o_1$ to $\o_2$ by a path in $\Sigma \cap K$. This contradicts that z_1 and z_2 are in distinct components of $\Sigma - M$. Hence M intersects K.

To complete the proof of the theorem we will show that when $p \in \Sigma - K$ is far enough from K then p can not have a vertical tangent plane. We will do this by showing such a vertical tangent plane can not intersect K.

To do this we introduce comparison surfaces $M(h)$, first introduced by Hauswirth [9], then by Toubiana and Sa Earp and Toubiana [14], Daniel [6] and Mazet, Rodriguez, Rosenberg [15]. The surfaces $M(h)$ are all congruent in $\mathbb{H} \times \mathbb{R}$, they are complete minimal surfaces invariant by hyperbolic translation along a geodesic of $\mathbb{H}$. They exist for each $h > \pi$; we state the properties we will use.

(1) Let β be an equidistant of a geodesic γ of $\mathbb{H} \times \{0\}$, whose distance d_0 to γ is determined by h. There is an $M(h)(= M(h, \beta))$ which is a minimal bigraph over the domain Ω of $\mathbb{H}$ indicated in figure 6.
(2) $M(h)$ has height $h/2$ over $\Omega \times \{0\}$,

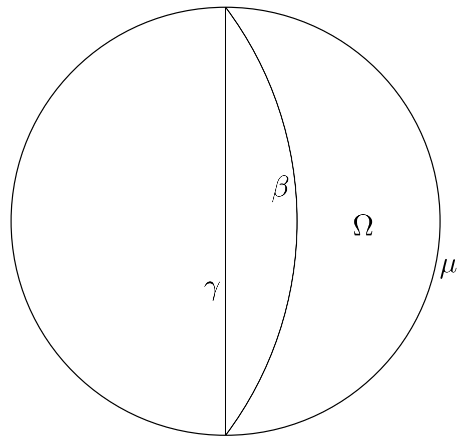

FIGURE 6. Domain Ω

(3) $M(h)$ meets $\mathbb{H} \times \{0\}$ orthogonally along β,
(4) $M(h)$ meets each $\mathbb{H} \times \{t\}$ in an equidistant of $\gamma \times \{t\}$ for $|t| < h/2$,
(5) The asymptotic boundary of $M(h)$ is the vertical rectangle of $\mathbb{H} \times \mathbb{R}$:

$$\partial_\infty(M(h)) = (\partial_\infty(\beta) \times [-h, h]) \cup (\mu \times \{\pm h/2\}).$$

μ is the arc in $\partial_\infty(\mathbb{H})$ joining the end points of β, indicated in figure 6.
(6) Given $q \in \mathbb{H}$ and a vector v tangent to $\mathbb{H}$ at q, there exists an $M(h, \beta) = M(h)$ such that $q \in \beta$ and the tangent to β at q is orthogonal to v.
(7) Now assume $h \geq 2\pi$ and S is the slab of height $\pi - \epsilon : S = \{|t| \leq (\pi - \epsilon)/2\}$. Then there exists $d_2 > 0$ such that if $|t| < (\pi - \epsilon)/2$ and T_t is a vertical translation by t then $\mathrm{dist}(T_t(M(h)) \cap S, \gamma \times \mathbb{R}) \leq d_2$. Here γ is the geodesic of which β is the equidistant. In fact, in any strict subslab $\tilde{S}$ of $\mathbb{H} \times] - h/2, h/2[$, $\tilde{S}$ symmetric about $t = 0$, $M(h)$ is a bigraph over the moon between β and another equidistant $\tilde{\beta}$ of γ and this moon is a bounded distance d_2 from γ.
(8) If γ_1 and γ_2 are complete geodesics of $\mathbb{H}$ then $M_{\gamma_1}(h)$ is congruent to $M_{\gamma_2}(h)$ by a height preserving isometry of $\mathbb{H} \times \mathbb{R}$.
(9) Let $p_0 = (0, y_0)$ and for $0 < y < y_0$, let β be an equidistant curve of γ such that β is tangent at $(0, y)$ to the geodesic α through $(0, y)$ and p_0 (here β is an equidistant of an $M(h)$). Then $\mathrm{dist}_{\mathbb{H}}(p_0, \gamma) \to \infty$, as $y \to 0$; see figure 7.

Here is one way to verify this last assertion. For $(0, y)$ and $h \geq 2\pi$ fixed, β, γ and M_γ are uniquely determined so that M_γ is tangent to $\alpha \times \mathbb{R}$ at $(0, y)$ at a point of β.

So it suffices to fix $(0, y)$, and let $p_l = l p_0$, with $l \to \infty$. The minimizing geodesic from p_l to γ tends to the vertical geodesic going up from the point of maximum y coordinate on γ; see figure 7. Hence it's length goes to ∞ as $l \to \infty$; see figure 7.

Now we are ready to finish the proof of the theorem. Let $C > 0$ be such that if $\mathrm{dist}_{\mathbb{H}}(p_0, (0, y)) \geq C$, then $\mathrm{dist}_{\mathbb{H}}(p_0, \gamma) > d_2 + \mathrm{diam}(K)$. Recall that p_0 is the center of the ball B. We write $p_0 = (0, y_0)$. We will prove that a point of Σ at a distance at least C from p_0, can not have a vertical tangent plane; this will prove the theorem.

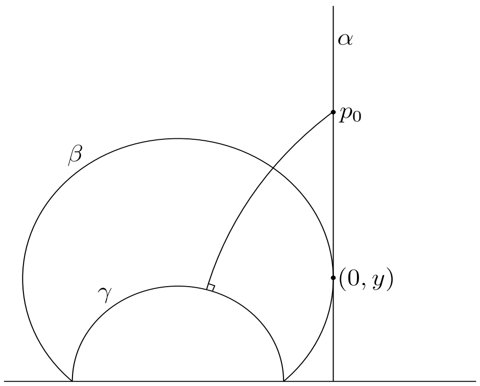

FIGURE 7. Minimizing geodesic from p to γ

Assume the contrary. Write $p=(0,y,t)$, $p\in\Sigma$ is a point with a vertical tangent plane and $\mathrm{dist}_{\mathbb{H}}(p_0,(0,y))\geq C$. We vertically translate $M(h)$ by some t, $|t|<\pi/2$, so that $M_t(h)=T_t(M(h))$ is tangent to Σ at p. Let γ be the geodesic associated to $M_t(h)$. Since $\mathrm{dist}_{\mathbb{H}}(p_0,(0,y))\geq C$, we know that $\mathrm{dist}_{\mathbb{H}}(p_0,\gamma)>d_2+\mathrm{diam}(K)$.

Now if $M_t(h)$ does not intersect K, the same proof we gave using $M_t(h)$ in place of $M=\alpha\times\mathbb{R}$, shows we obtain a contradiction.

We explain this further. Let η be a geodesic of $\mathbb{H}$ orthogonal to γ, $p\in\eta$. The set of all geodesics $\gamma(s)$, $s\in\mathbb{R}$, of $\mathbb{H}$, orthogonal to η, foliates $\mathbb{H}$ and γ is a leaf of this foliation For $h>\pi$, the $M(h,\gamma(s))$ foliate the slab of $\mathbb{H}\times\mathbb{R}$ between $t=h$ and $t=-h$. Hence if D is a compact minimal surface whose boundary is in some $M(h,\gamma(s))$, then D is contained in $M(h,\gamma(s))$. The other property of this foliation one uses is the following: If Σ is a properly immersed minimimal surface in the slab S (height $\pi-\epsilon$) and the intersection of Σ and $M(h,\gamma(\epsilon))$ has a non compact component C, for some $\epsilon>0$, then $\mathrm{dist}(C,M(h,\gamma(0)))$ tends to infinity as one diverges in C.

Hence it suffices to show $M_t(h)$ can not intersect K. Suppose there is a point $w\in M_t(h)\cap K$. Then $\mathrm{dist}(\omega,p_0)\leq\mathrm{diam}K$, and $\mathrm{dist}(\omega,\gamma)\leq d_2$, so $\mathrm{dist}(p_0,\gamma)\leq d_2+\mathrm{diam}K$; a contradiction.

Now suppose Σ of the theorem is embedded. Let $r>0$ and $C(r)=\{p\in\mathbb{H};d(p,p_0)=r\}$. Define $\mathrm{Cyl}(r)=C(r)\times\mathbb{R}$; $\mathrm{Cyl}(r)$ is a vertical cylinder of radius r.

Let $r_0>0$ be large so that Σ is a multigraph for $r\geq r_0$ (we proved Σ is not vertical for large r). Σ is proper, so $\Sigma\cap\mathrm{Cyl}(r)$ is a finite union of embedded Jordan curves, each a graph over $C(r)$, for each $r\geq r_0$.

Let $\beta(r)$ be one of the graphical components of $\Sigma\cap\mathrm{Cyl}(r)$, $r\geq r_0$. If Σ is simply connected (so $\partial\Sigma=\emptyset$), then the usual proof of Rado's theorem shows $\beta(r)$ bounds a unique compact minimal surface that is a graph over the disk bounded by $C(r)$. Hence Σ is an entire graph.

The same arguments shows that if $\partial\Sigma=\emptyset$ and Σ has an annular end than $\Sigma\cap\mathrm{Cyl}(r)$ is one Jordan curve, a graph over $C(r)$, that (by Rado's theorem) bounds a unique

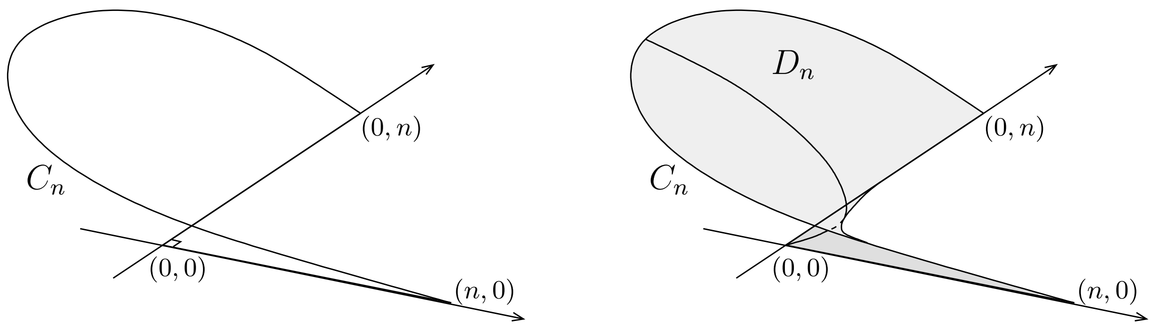

FIGURE 8. Left: A Jordan curve C_n. Right: An Enneper's sector D_n

compact minimal surface, a graph over the disk bounded by $C(r)$. This proves (1) of the Slab theorem.

Now suppose Σ is a properly immersed minimal annulus in S, with compact boundary. Choose r large so that $\partial\Sigma \subset \mathrm{Cyl}(r)$ and $\Sigma \cap \mathrm{Cyl}(r)$ is a finite union of multi-graphs. Since Σ has one end, there is exactly one multi-graph, and the end is a multi-graph. If Σ is embedded $\Sigma \cap \mathrm{Cyl}(r)$ is a graph and the end is a graph. This proves (2).

Remark 3.1. *We now describe the surface discussed in Remark 1.2. Consider the surface $M_\gamma(h), h > \pi$ and γ a geodesic of $\mathbb{H}$. Fix a point p in γ and deform γ through equidistant curves $\beta(t)$, $0 \le t \le 1$, through p, such that $\beta(0) = \gamma$ and $\beta(1)$ is a horocycle (do this by continuously deforming the endpoints of γ to make them converge to one point at infinity). Then the surfaces $M_{\beta(t))}(h(t))$ converge to a minimal surface Σ of height π. The asymptotic boundary of Σ is a vertical segment of height π. Σ is vertical along $\beta(1)$, this simply connected embedded minimal surface shows the Slab Theorem fails for slabs of height π. Benoit Daniel explicitly parametrized this surface; cf proposition 4.17 of* [7].

4. AN ENNEPER TYPE MINIMAL SURFACE

We construct a properly immersed simply connected minimal surface in $\mathbb{H} \times \mathbb{R}$ contained in a slab S (of any height h) that is a 3-sheeted multi-graph outside of a compact set.

The idea comes from Enneper's minimal surface $\mathcal{E}$ in $\mathbb{R}^3$, whose Weierstrass data is $(g, \omega) = (z, dz)$ on the complex plane $\mathbb{C}$. We think of $\mathcal{E}$ as constructed by solving Plateau problems for certain Jordan curves in $\mathbb{R}^3$; passing to the limit, and then reflecting about the two rays in the boundary. More precisely let C_n be a Jordan curve consisting of the segments on the x and y axes between 0 and n, and the (slightly tilted up) large arc on the circle of radius n (centered at $(0,0)$) joining $(n,0)$ to $(0,n)$); see figure 8.

Let D_n be a disk of minimal area with $\partial D_n = C_n$; see figure 8-Right.

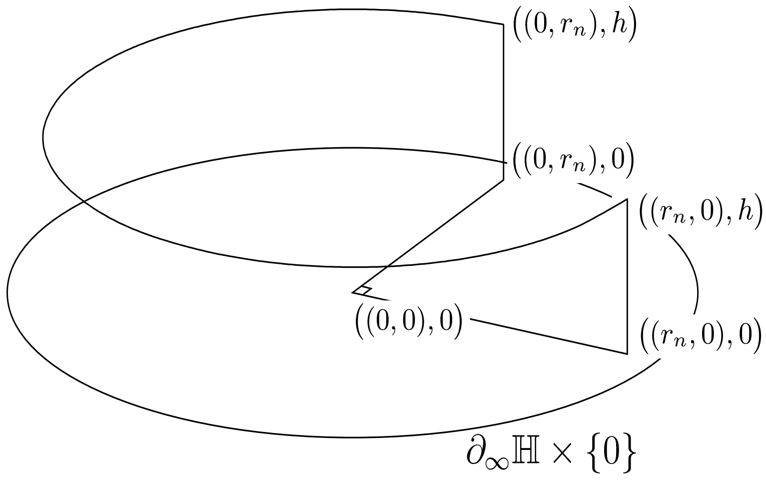

FIGURE 9. A Jordan curve in $\mathbb{H}\times\mathbb{R}$

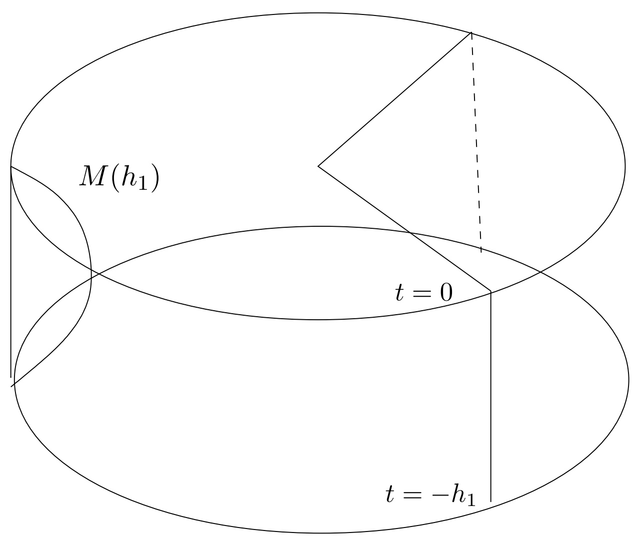

FIGURE 10. The barrier $M(h_1)$

One can choose C_n so the D_n converge to a minimal surface (not flat) with boundary the positive x and y axes. $\mathcal{E}$ is then obtained by Schwarz reflection in the x and y axes. We now do this in $\mathbb{H}\times\mathbb{R}$.

We use now the unit disk $\{x^2+y^2<1\}$ with the hyperbolic metric as a model for $\mathbb{H}$. Let $h>0$ and n an integer, $n\geq 1$. In $\mathbb{H}\times\mathbb{R}$, let C_n be the Jordan curve; see figure 9:

$$\begin{aligned} C_n = &\{(x,0); 0 \le x \le r_n\} \cup \{(0,y); 0 \le y \le r_n\} \\ &\cup \{(r_n \cos\theta, r_n \sin\theta, h); \pi/2 \le \theta \le 2\pi\} \\ &\cup ((r_n, 0) \times [0,h]) \cup ((0, r_n) \times [0,h]) \end{aligned}$$

where $r_n \to 1$. Let D_n be a least area minimal disk with $\partial D_n = C_n$. We claim a subsequence of D_n converges to a stable non trivial minimal surface with boundary the positive x and y axes. Σ is then obtained by reflection in the horizontal boundary geodesics.

First observe the C_n converge in the model of the disk to

$$\begin{aligned} C = &\{(x,0); 0 \le x \le 1\} \cup \{(0,y); 0 \le y \le 1\} \\ &\cup \{(\cos\theta, \sin\theta, h); \pi/2 \le \theta \le 2\pi\} \\ &\cup ((1,0) \times [0,h]) \cup ((0,1) \times [0,h]) \end{aligned}$$

The latter circular arc at infinity- $\{(\cos\theta, \sin\theta, h); \pi/2 \le \theta \le 2\pi\}$ is the upper part of a vertical rectangle at infinity that bounds a minimal surface $M(h_1)$, $h_1 > \pi$. We remark that as $h_1 \to \pi$, $h_1 > \pi$ the β of $M(h_1)$ tends to a horocycle. Hence $d_2 = \mathrm{dist}(\beta, \gamma) \to \infty$ as $h_1 \to \pi$. This implies $M(h_1)$ is a barrier for each of the D_n, for h_1 sufficiently close to π; see figure 10.

Translate $M(h_1)$ down to be below height 0, then go back up to height h. There can be no contact with D_n. To prevent D_n from escaping in the sector $0 \le \theta \le \pi/2$, one can place another $M(h_1)$ in this sector, whose vertical rectangle at infinity has horizontal arc of angle less than $\pi/2$ and whose vertical sides go from a height below zero to a height above h.

Let F be the vertical solid cylinder in $\mathbb{H} \times \mathbb{R}$ over the circle of radius 1 in $\mathbb{H}$, centered at $(-2,-2)$ (hyperbolic distances). For $n > 2$, each D_n intersects F in a minimal surface of bounded curvature (bound independent of n). Hence a subsequence of these surfaces converges to a minimal surface, which is not part of a horizontal slice. The barriers $M(h_1)$ then show a subsequence of the D_n converge to a minimal surface Σ_1 with boundary the positive x and y axes. To guaranty that D_n does not escape in the sector $\{0 \le \theta \le \pi/2\}$, one puts a barrier $M(h_2)$ as in figure 11.

Now do the reflection about the positive x and y axis and then about the negative x and y axes. This gives a complete minimal (Enneper type) immersed surface. One checks the origin is not a singularity as follows. Reflecting D_n four times gives an immersed minimal punctured disk that has the origin in its closure. Gulliver [8] then proved the origin is not a singularity.

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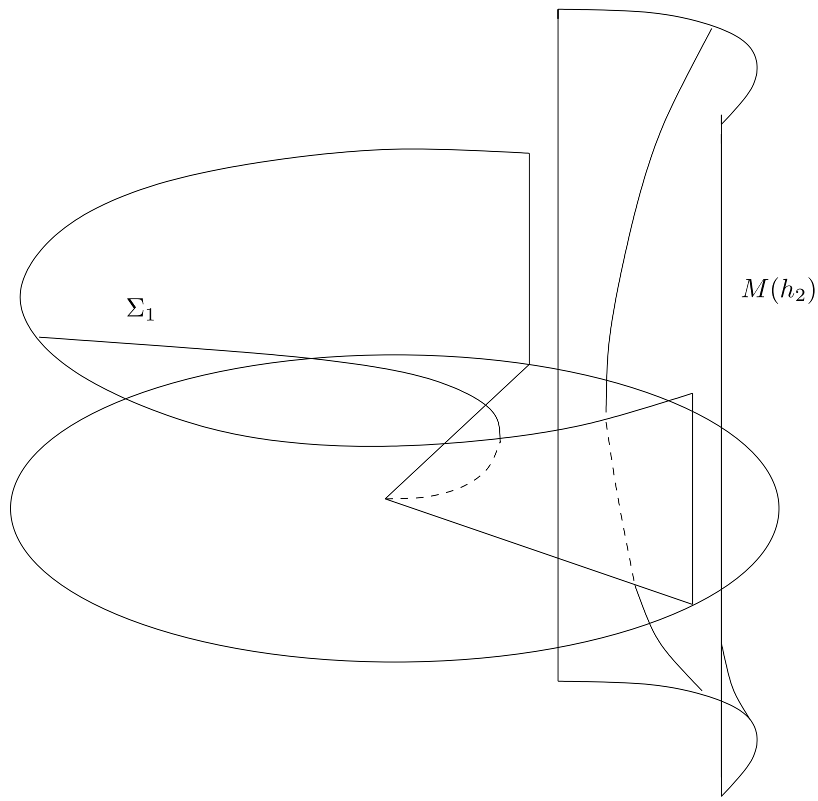

FIGURE 11. The barrier $M(h_2)$

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